IUCAA GR Refresher Course Tutorials by BM

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Tutorial:4 (26-06-2025)

Problem:1 Ricci Tensor Components in FRW Spacetime

The **Friedmann–Robertson–Walker (FRW) metric**, which describes a homogeneous and isotropic expanding universe, is given by:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right],$$

where $a(t)$ is the scale factor, and $k = 0, \pm 1$ denotes the spatial curvature (flat, closed, or open universe).

Tasks:

- Compute the non-zero components of the Ricci tensor $R_{\mu\nu}$ for the above FRW metric using the Christoffel symbols.
- Express the Ricci tensor components in terms of the scale factor $a(t)$ and its derivatives $\dot{a}(t)$, $\ddot{a}(t)$.
- Interpret the physical meaning of the R_{00} and spatial components R_{ij} in the context of cosmic expansion.

Problem:2 Energy-Momentum Tensor in a Homogeneous and Isotropic Universe

In General Relativity, the energy-momentum tensor for a **perfect fluid** is given by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$$

where: $\rho(t)$ is the energy density, $p(t)$ is the pressure and u^μ is the four-velocity of the fluid (with $u^\mu = (1,0,0,0)$ in comoving coordinates).

In a universe described by the **Friedmann–Robertson–Walker (FRW) metric**:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right],$$

the universe is assumed to be **homogeneous and isotropic**.

<u>Tasks:</u>

1. Using the form of $T_{\mu\nu}$ for a perfect fluid, identify the **non-zero components** of the energy-momentum tensor in comoving coordinates. Show that:

$$T_{00} = \rho(t), \quad T_{ij} = p(t)\, g_{ij}, \quad T_{0i} = 0$$

2. Write down the **covariant conservation law**:

$$\nabla^\mu T_{\mu\nu} = 0$$

and compute the $\nu = 0$ component explicitly. Derive the **continuity equation**:

$$\dot{\rho} + 3H(\rho + p) = 0,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

3. Discuss how this conservation equation is **not independent** of the Friedmann equations, and how it can be derived from them using the Einstein field equations.